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NOTE ON INTEGRAL AND INTEGRO-GEOMETRIC SERIES.*

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PREFACE.

So far as I know, the methods which have been given for summing the integral and integro-geometric series are, with perhaps the possible exception of the sum of the powers of the natural numbers, methods which give directions how to proceed in a particular case, but offer no completed general solutions of the summations. For the integral series the best method directs to the use of a combination of undetermined coefficients, which are to be obtained for each particular problem, with scales of relation for the sum of the powers of the natural numbers, or else with repeated calculations of the sums of the powers of the natural numbers. For the integro-geometric series a multiplication by a power of $(1 - x)$ is suggested. It is made plain that these processes will succeed in a given case. These methods are well exemplified in Chrystal's Algebra, Pt. I, pp. 107, 484–493, and 504 Ex. 47.

This lack of general solutions led me to inquire after their possibility and the form which they might assume. Complete general formulas were obtained.

The investigation led to a series of numbers to whose study I am not aware that attention has been heretofore particularly directed. In themselves they possess considerable interest, and by means of them or their properties the summations are effected. They are coefficients in an exponential expansion. It is believed that this article is the first to call especial attention to them, and to establish their existence in the general formula for the sum of an integral series, and their relation to that for the integro-geometric series.

The investigation is also characterized by the generous use of the method of finite differences, which was found to be of great service.

The work is divided into three parts: First, the consideration of the numbers, called here $A_{r+\lambda}$; second, the development of a function by finite differences, and its application to the integral series; and, third, the summations of the integral and integro-geometric series.

I. The numbers $A_{r+\lambda}$.

A. Definition. $A_{r+\lambda}$ is the coefficient of $x^{r+\lambda}$ in $(e^x - 1)^r$. It will be denoted usually by $A_{r+\lambda}$, but when ambiguity might arise it will be distinguished by $A_{(r)+\lambda}$. We have $A_{(r)+\lambda} \equiv A_{r+\lambda}$.

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* Presented at the summer meeting of the American Mathematical Society at Buffalo, 1st September, 1896.

B. Formation and Elementary Properties.

a. By expansion of $(e^x - 1)^r$, $A_{r+\lambda}$ is found to be equal to

$$\frac{1}{(r+\lambda)!}\left[r^{r+\lambda} - r(r-1)^{r+\lambda} + \frac{r \, . \, r-1}{1 \, . \, 2}(r-2)^{r+\lambda} - \ldots (-1)^{r-1}r\right].$$

It is easily found that $A_r = 1$, $A_{r+1} = \frac{r}{2}$, $A_{r+2} = \frac{r}{3!} + \frac{r(r-1)}{8}$, and it will appear in its summation, *C*, *b*, that $A_{r+\lambda}$ is a rational, integral, and algebraic function of the λth degree in r.

b. The function

$$n^k - r(n-1)^k + \frac{r \, . \, r-1}{1 \, . \, 2}(n-2)^k - \ldots (-1)^r(n-r)^k = f(n)$$

is an integral function of n of the $(k-r)$th degree, and vanishes if $k < r$, or if $k = r + \lambda$, vanishes if $\lambda < 0$.

This may be proved as follows:

The function $n^k - (n-1)^k$ is of degree $(k-1)$ in n.

The function $(n-1)^k - (n-2)^k$ is of degree $(k-1)$ in n.

$\therefore$ the function $n^k - (n-1)^k - ((n-1)^k - (n-2)^k)$, or $n^k - 2(n-1)^k + (n-2)^k$ is of degree $k-2$ in n. By mathematical induction, and by means of the relation $C_{n,r-1} + C_{n,r} = C_{n+1,r}$ it is then easily proved that

$$n^k - r(n-1)^k + \frac{r \, . \, r-1}{1 \, . \, 2}(n-2)^k - \ldots (-1)^r(n-r)^k$$

is of degree $(k-r)$ in n.

It follows that

$$n^k - k(n-1)^k + \frac{k \, . \, k-1}{1 \, . \, 2}(n-2)^k - \ldots (-1)^k(n-k)$$

is of degree 0 in n, i. e., is constant, and independent of the value of n. If in this last expression $n = k$, we obtain

$$k^k - k(k-1)^k + \frac{k \, . \, k-1}{1 \, . \, 2}(k-2)^k - \ldots (-1)^{k-1}k = k!\,A_k = k!\,,$$

and hence

$$n^k - k(n-1)^k + \frac{k \, . \, k-1}{1 \, . \, 2}(n-2)^k - \ldots (-1)^k(n-k) = k!\,,$$

and because this function is constant, if we give n the value $n-1$, we get

$$(n-1)^k - k(n-2)^k + \frac{k \, . \, k-1}{1 \, . \, 2}(n-3)^k - \ldots (-1)^k(n-k-1)^k = k!\,.$$

Subtracting this expression from the previous one and reducing by means of the relation $C_{n,r-1} + C_{n,r} = C_{n+1,r}$, we get

$$n^k - (k+1)(n-1)^k + \frac{k+1\,.\,k}{1\,.\,2}(n-2)^k - \ldots (-1)^{k+1}(n-k-1)^k \equiv 0\,.$$

And, in a similar way, by mathematical induction we can obtain

$$n^k - (k+s)(n-1)^k + \frac{k+s\,.\,k+s-1}{1\,.\,2}(n-1)^k$$
$$- \ldots (-1)^{k+s}(n-k-s)^k \equiv 0\,,$$

i. e., vanishes when $k - (k+s) = -s$, which is what we took for λ. Thus $f(n)$ vanishes when $\lambda < 0$.*

c. If in $f(n)$ we put $k = r + \lambda$, we get by Maclaurin's development, the details of which need not be given here,

$$f(n) = (r+\lambda)!$$
$$\left[\frac{A_r n^\lambda}{\lambda!} - \frac{A_{r+1}n^{\lambda-1}}{(\lambda-1)!} + \frac{A_{r+2}n^{\lambda-2}}{(\lambda-2)!} - \ldots (-1)^p \frac{A_{r+p}n^{\lambda-p}}{(\lambda-p)!} \ldots + (-1)^\lambda A_{r+\lambda}\right]$$

$f(n)$ may also be looked upon as $k!$ × (the coefficient of x^k) in

$$e^{nx} - re^{(n-1)x} + \frac{r\,.\,r-1}{1\,.\,2}e^{(n-2)x} - \ldots$$
$$= e^{(n-r)x}\left[e^{rx} - re^{(r-1)x} + \frac{r\,.\,r-1}{1\,.\,2}e^{(r-2)x} - \ldots\right]$$
$$= e^{(n-r)x}(e^x - 1)^r$$
$$= \left[1 + (n-r)x + \frac{(n-r)^2x^2}{2!} + \frac{(n-r)^p x^p}{p!} + \right](e^{rx} - re^{(r-1)x} + \ldots)$$

and hence

$$f(n) = (r+\lambda)!$$
$$\left[\frac{A_r(n-r)^\lambda}{\lambda!} + \frac{A_{r+1}(n-r)^{\lambda-1}}{(\lambda-1)!} + \ldots \frac{A_{r+p}(n-r)^{\lambda-p}}{(\lambda-p)!} \ldots + A_{r+\lambda}\right].$$

If $n = r$ in the two forms, we obtain by equating them,

$$(1 - (-1)^\lambda)A_{r+\lambda} = \sum_{p=0}^{p=\lambda-1}(-1)^p \frac{A_{r+p}r^{\lambda-p}}{(\lambda-p)!}\,,$$

a relation which would certainly give all odd λ subscript coefficients when the preceding ones both odd and even are given. It gives no new information, however, if λ is even. The complete solution for the A's will be given farther along.

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* This theorem has been obtained by Chrystal in a different way. Part II, p. 183 (3) p. 210. To make this give his result put $n = x + n'$, $k = s'$, $s = n' - s'$ and drop accents.

d. That the function $A_{r+\lambda}$ is of degree λ in r, will appear later from its summation, and may be partially seen here as follows: In $(e^x - 1)^r$, it is clear that the coefficient of $x^{r+\lambda}$ is $(r > \lambda)$

$$\frac{r}{(\lambda + 1)!} + \frac{r\,.\,r - 1}{1\,.\,2}\,a_2 + \frac{r\,.\,r - 1\,.\,r - 2}{1\,.\,2\,.\,3_{[a_{\lambda-1}]}}\,a_3 + \ldots$$

$$\frac{r\,.\,r - 1\,.\,r - 2 \ldots x - \lambda + 2}{(\lambda - 1)!}\left[\frac{\lambda - 1}{(2\,!)^{\lambda-4}_{[a_\lambda]}}\frac{1}{A\,!} + \frac{(\lambda - 1)(\lambda - 2)}{(2\,!)^{\lambda-2}}\frac{1}{3\,!}\right]$$

$$+ \frac{r\,.\,r - 1\,.\,r - 2 \ldots r - \lambda + 1}{\lambda!}\left[\frac{1}{2\,!}\right]^{\lambda}.$$

The last term of degree λ in r containing one term of degree $r + \lambda$ in x, comes from

$$\frac{r\,.\,r - 1 \ldots r - \lambda + 1}{\lambda\,!}\,x^{r-\lambda}\left[\frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots\right]^{\lambda},$$

and the next term

$$\frac{r\,.\,r - 1 \ldots r - \lambda}{(\lambda + 1)!}\,x^{r-\lambda-1}\left[\frac{x^2}{2\,!} + \ldots\right]^{\lambda+1}$$

contains no term $x^{r+\lambda}$, the lowest in x being of degree $r - \lambda - 1 + 2\lambda + 2 = r + \lambda + 1$. These results come from considering $(e^x - 1)^r$ in the form of

$$\left[x + \left[\frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots\right]\right]^r.$$

e. By the method of differences, to be noticed farther along, it is given that, $(r > \lambda)$

$$A_{(r)+\lambda} = A_{(0)+\lambda} + r(A_{(1)+\lambda} - A_{(0)+\lambda})$$

$$+ \frac{r\,.\,r - 1}{1\,.\,2}(A_{(2)+\lambda} - 2A_{(1)+\lambda} + A_{(0)+\lambda}) + \ldots$$

$$+ \frac{r\,.\,r - 1 \ldots r - \lambda + 1}{1\,.\,2 \ldots \lambda}$$

$$(A_{(\lambda)+\lambda} - \lambda A_{(\lambda-1)+\lambda} + \frac{\lambda\,.\,\lambda - 1}{1\,.\,2}A_{(\lambda-2)+\lambda} - (-1)^{\lambda}A_{(0)+\lambda})$$

with the last term as here given, otherwise $A_{(r)+\lambda}$ would be of higher degree than the λth, which by d is impossible. (Cf. C. b.)

By referring to C. b. the restriction $r > \lambda$ is seen to be removed.* Comparing the two forms of e, and d, and putting $r = 0, 1, 2, \ldots$ in them successively, we get ($\lambda > 0$),

$$a_0 = A_{(0)+\lambda} = 0$$

$$a_1 = A_{(1)+\lambda} - A_{(0)+\lambda} = \frac{1}{(\lambda+1)!}$$

$$a_2 = A_{(2)+\lambda} - 2A_{(1)+\lambda} + A_{(0)+\lambda} = \frac{2^{2+\lambda}-2}{(2+\lambda)!} - \frac{2}{(1+\lambda)!}$$

. . . .

$$a_{\lambda-1} = \frac{\lambda-1}{(2!)^{\lambda-2}}\cdot\frac{1}{4!} + \frac{(\lambda-1)(\lambda-2)}{(2!)^{\lambda-2}}\,\frac{1}{3!}$$

$$a_\lambda = A_{(\lambda)+\lambda} - \lambda A_{(\lambda-1)+\lambda} + \frac{\lambda\,.\,\lambda-1}{1\,.\,2}A_{\lambda-2+\lambda} - \ldots (-1)^\lambda A_{0)+\lambda} = \left[\frac{1}{2!}\right]^\lambda .$$

It also follows that, $k > 0$,

$$A_{(\lambda+k)+\lambda} - (\lambda+k)\,A_{(\lambda+k-1)+\lambda}$$
$$+ \frac{(\lambda+k)(\lambda+k-1)}{1\,.\,2}A_{(\lambda+k-2)+\lambda} - \ldots (-1)^{\lambda+k} A_{(0,+\lambda} \equiv 0\,.$$

As a special case if $k = 1$, $\lambda + 1 = r$,

$$A_{(r)+r-1} - rA_{(r-1)+r-1} + \frac{r\,.\,r-1}{1\,.\,2}A_{(r-2)+r-1} - \ldots (-1)^r A_{(0)+r-1} \equiv 0\,.$$

f. Again by the method of finite differences,

$$A_{r+\lambda} = A_r + \lambda\,(A_{r+1} - A_r) + \frac{\lambda\,.\,\lambda-1}{1\,.\,2}(A_{r+2} - 2A_{r+1} + A_r) + \ldots$$
$$+ \frac{\lambda\,.\,\lambda-1\,\ldots\,\lambda-r+1}{1\,.\,2\,\ldots\,r}(A_{(r)+r} - rA_{(r)+r-1} + \ldots) + \ldots$$

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* It is true that when $\lambda = r + s$, where s may be 0, 1, 2, . . . , λ, the function $A_{r+\lambda}$ may be written as of degree r in r, as well as of degree λ in r, for the particular value $r = \lambda - s$. E. g.: $A_{r+2} = \frac{r}{6} + \frac{r(r-1)}{8}$. Here $\lambda = 2$. Let $s = 1$, then $r = 1$, and for $r = 1$,

$$\frac{r}{6} = \frac{r}{6} + \frac{r(r-1)}{8} = \frac{r^2}{8} + \frac{r}{24}\,.$$

which shows that $A_{r+\lambda}$ may be expressed as of the λth degree in λ, or in any higher degree for a particular value of λ.

Let $r = 2$, and

$$A_{(2)+\lambda} = A_2 + \lambda\,(A_{(2)+1} - A_2 + \frac{\lambda\,.\,\lambda - 1}{1\,.\,2}\,(A_{(2)+2} - 2A_{(2)+1} + A_2) + \ldots,$$

or

$$\frac{2^{2+\lambda} - 2}{(2+\lambda)} = \frac{2^2 - 2}{2\,!} + \lambda\left[\frac{2^3 - 2}{3\,!} - \frac{2^2 - 2}{2\,!}\right]$$

$$+ \frac{\lambda\,(\lambda - 1)}{1\,.\,2}\left[\frac{2^4 - 2}{4\,!} - 2\,\frac{2^3 - 2}{3\,!} + \frac{2^2 - 2}{2\,!}\right] + \cdots$$

or

$$\frac{2^{2+\lambda} - 2}{(2+\lambda)\,!} = 1 + \frac{5}{2A}\,\lambda - \frac{5}{2A}\,\lambda^2 + \ldots$$

a curious relation in that the equation is absolutely true to three terms if $\lambda = 0, 1, 2$, as may be seen by substitution, but is not generally true in this way for all values of λ; similarly $\frac{1}{(\lambda + 1)\,!} = 1 - \frac{1}{2}\,\lambda$, is true if $\lambda = 0, 1$, but not generally. Other illustrations might be given.

C. The General Computation of the Numbers $A_{r+\lambda}$.

a. A scale of relation may be found for these numbers as follows :

$$fx = (e^x - 1)^r = A_r x^r + A_{r+1}x^{r+1} + \ldots = \sum_{\lambda=0}^{\lambda=\infty} A_{r+\lambda}x^{r+\lambda}\,.$$

$$f'x = r\,(e^x - 1)^{r-1}\,e^x = \sum_{\lambda=0}^{\lambda=\infty} (r + \lambda)\,A_{r+\lambda}x^{r+\lambda-1}\,.$$

Now $(e^x - 1)\,f'x - re^x fx \equiv 0$, or

$$\left[\sum_{\lambda=0}^{\lambda=\infty} (r + \lambda)\,A_{r+\lambda}x^{r+\lambda-1}\right]\left[\sum_{\lambda=1}^{\lambda=\infty} \frac{x^\lambda}{\lambda!}\right] - r\left[\sum_{\lambda=0}^{\lambda=\infty} \frac{x^\lambda}{\lambda!}\right]\left[\sum_{\lambda=0}^{\lambda=\infty} A_{r+\lambda}x^{r+\lambda}\right] \equiv 0\,.$$

In this equation the coefficient of $x^{r+\lambda}$ is 0 ; it is

$$\left[\frac{(r + \lambda)}{1!}\,A_{r+\lambda} + \frac{(r + \lambda - 1)}{2\,!}\,A_{r+\lambda-1} + \frac{r + \lambda - 2}{3\,!}\,A_{r+\lambda-2} + \ldots \frac{rAr}{(\lambda + 1)!}\right]$$

$$- r\left[\frac{A_{r+\lambda}}{0!} + \frac{A_{r+\lambda-1}}{1!} + \frac{A_{r+\lambda-2}}{2\,!} + \ldots \frac{rAr}{\lambda!}\right],$$

and we have

$$\lambda A_{r+\lambda} + \left[\frac{r+\lambda-1}{2!} - \frac{r}{1!}\right] A_{r+\lambda-1} + \left[\frac{r+\lambda-2}{3!} - \frac{r}{2!}\right] A_{r+\lambda-2}$$
$$+ \dots \left[\frac{r}{(\lambda+1)!} - \frac{r}{\lambda!}\right] A_r = 0 .$$

This gives a scale of relation for computing the numbers $A_{r+\lambda}$.

b. The complete solution for $A_{r+\lambda}$ may however be expressed in the form of a determinant. For, similarly to the preceding scale, we have by changing λ into $\lambda-1, \lambda-2, \dots, 1$,

$$(\lambda-1) A_{r+\lambda-1} + \left[\frac{r+\lambda-2}{2!} - \frac{r}{1!}\right] A_{r+\lambda-2} + \dots \left[\frac{r}{\lambda!} - \frac{r}{(\lambda-1)!}\right] A_r = 0 .$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A_{r+1} + \left[\frac{r}{2!} - \frac{r}{1!}\right] A_r = 0 .$$

These together with the scale of a. give λ equations. Solving them for $A_{r+\lambda}$, we get after substituting for A_r its value 1, and noticing that the denominator of the solution

$$\begin{vmatrix} \lambda & \frac{r+\lambda-1}{2!} - \frac{r}{1!} & \dots & & \left[\frac{r}{(\lambda+1)!} - \frac{r}{\lambda!}\right] \\ 0 & \lambda-1 & \dots & & \\ 0 & 0 & \lambda-2 \dots & & \\ \dots & \dots & \dots & \dots & \dots \\ & & & 2 & \\ 0 & 0 & 0 \dots & 0 & 1 \end{vmatrix} = \lambda!$$

$$A_{r+\lambda} = \frac{1}{\lambda!} \begin{vmatrix} \frac{r}{\lambda!} - \frac{r}{(\lambda+1)!} & \frac{r+\lambda-1}{2!} - \frac{r}{1!} & \frac{r+\lambda-2}{3!} - \frac{r}{2!} & \dots & \frac{r+1}{\lambda!} - \frac{r}{(\lambda-1)!} \\ \frac{r}{(\lambda-1)!} - \frac{r}{\lambda!} & (\lambda-1) & \frac{r+\lambda-2}{2!} - \frac{r}{1!} & \dots & \frac{r+1}{(\lambda-1)!} - \frac{r}{(\lambda-2)!} \\ & 0 & \lambda-2 & \dots & \\ & 0 & 0 & \dots & \\ \frac{r}{1!} - \frac{r}{2!} & 0 & 0 & \dots & \frac{r+1}{1!} - \frac{r}{0!} \end{vmatrix}$$

By reducing the form of the elements, and writing as a cyclical determinant, as I shall call it,

$$A_{r+\lambda} = \frac{(-1)^\lambda}{\lambda\,!} \begin{vmatrix} \frac{\lambda-(r+1)}{2\,!} & \frac{\lambda-2\,(r+1)}{3\,!} & & \cdots & \frac{\lambda-\lambda\,(r+1)}{(\lambda+1)\,!} \\ (\lambda-1) & \frac{\lambda-1-(r+1)}{2\,!} & & \cdots & \\ 0 & (\lambda-2) & & \cdots & \\ 0 & 0 & & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1\ \frac{1-(r+1)}{2\,!} \end{vmatrix}$$

It is clear that this determinant contains $2^{\lambda-1}$ terms of the complete number $\lambda\,!$ of the determinant of the λth order. It also appears at once from it that $A_{r+\lambda}$ is a rational, integral, and algebraic function of r of the λth degree. (v. B. d.)

II. The Development of a Function by Finite Differences.[*]

Let

$$\begin{array}{lll} f(x+0) = u_0 & u_1 - u_0 = \Delta u_0 & \Delta u_1 - \Delta u_0 = \Delta^2 u_0 \\ f(x+1) = u_1 & u_2 - u_1 = \Delta u_1 & \Delta u_2 - \Delta u_1 = \Delta^2 u_1 \\ f(x+2) = u_2 & u_3 - u_2 = \Delta u_2 & \text{and generally} \\ f(x+r) = u_r & u_r - u_{r-1} = \Delta u_{r-1} & \Delta^{r-1}_{u_1} - \Delta^{r-1}_{u_0} = \Delta^2 u^0 \\ f(x+n) = u_n & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \qquad = \Delta^r f(x)\,. \end{array}$$

Then by the method of differences, as special cases, we have :

A.
$$f(x+n) = f(x) + n\Delta f(x) + \frac{n\,(n-1)}{1\,.\,2}\Delta^2 f(x) + \ldots$$

$$= \sum_{r=0}^{r=n} C_{nr}\Delta^r f(x) = (1+\Delta)^n f(x) = D^n f(x)\,,$$

where the operator $1 + \Delta = D$.

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[*] Cf. Chrystal Algebra, Pt. II, pp. 372–379. Boole, Calculus of Finite Differences, pp. 13–19.

And also

B. $$\varDelta^n f(x) = f(x+n) - nf(x+n-1) + \frac{n\,.\,(n-1)}{1\,.\,2} f(x+n-2) + .\quad + (-1)^n f(x)\,.$$

$$= \sum_{v=0}^{r=n} (-1)^r C_{nr} f(x+n-r) = (D-1)^n f(x)\,.$$

Here $\varDelta = D - 1$, which shows that these operators obey the algebraic laws of addition and subtraction. As an example of these developments take

$$f(x+n) = a_0(x+n)^k + a_1(x+n)^{k-1} + \ldots a_k\,,$$

where the a's are constants. By A

$$f(x+n) = f(x) + n\varDelta f(x) + \frac{n(n-1)}{1\,.\,2}\varDelta^2 f(x) + \ldots \frac{n(n-1)\ldots(n-k+1)\,\varDelta^k fx}{1\,.\,2\ldots k}\,.$$

The development ends here because by I. B. b. $\varDelta^{k+1}f(x)$, $\varDelta^{k+2}f(x)$, etc., are equal to zero.

We can unite $f(x+n) = \sum_{r=0}^{r=k} C_{nr}\varDelta^r f(x)$. By B. of this division and by I. B. c. letting $n = x + r$ in the second form of $f(n)$ we get,

$$f(x+n) = \sum_{r=0}^{r=k} C_{nr} \sum_{\lambda=0}^{\lambda=k-r} a_{k-r-\lambda}(r+\lambda)!\sum_{p=0}^{p=\lambda}\frac{A_{r+p}x^{\lambda-p}}{(\lambda-p)!}$$

and this is the most general form of the nth, or general term of the integral series. If $x = 0$, we get,

$$a_0 n^k + a_1 n^{k-1} + \ldots a_k = \sum_{r=0}^{r=k} C_{nr} \sum_{\lambda=0}^{\lambda=k-r} a_{k-r-\lambda}(r+\lambda)!\,A_{r+\lambda}\,,$$

and this is the nth or general term of the integral series, as usually stated.

C. We have finally by the method of differences that the sum of the n terms $u_1, u_2, \ldots, u_n$ of this division is equal to

$$\sum_{r=0}^{r=k} C_{n+1\,r+1}\varDelta^r f(x) - u_0\,,$$

since for values of $r > k$, as already noted,

$$\varDelta^{k+s} f(x) = 0\,.$$

III. Summations.

A. Integral Series.

We have by II. B. and C.,

$$\sum_{r=1}^{r=n}{}' f(x+r) = \sum_{r=0}^{r=k}{}' C_{n+1\,r+1} \sum_{\lambda=0}^{\lambda=k-r}{}' a_{k-r-\lambda}(r+\lambda)! \sum_{p=0}^{p=\lambda}{}' \frac{A_{r+p}x^{\lambda-p}}{(\lambda-p)!}$$

$$-(a_0x^k + a_1x^{k-1} + \dots a_k),$$

and this is the most general formula for the sum of n terms of an integral series whose nth or general term is

$$f(x+n) = a_0(x+n)^k + a_1(x+n)^{k-1} + \dots a_k.$$

If $x = 0$, the sum becomes,

$$\sum_{r=0}^{r=k}{}' C_{n+1\,r+1} \sum_{\lambda=0}^{\lambda=k-r}{}' a_{k-r-\lambda}(r+\lambda)!\, A_{r+\lambda} - a_k.$$

If $a_1, \dots, a_k = 0$, $a_0 = 1$, we get

$$\sum_{r=1}^{r=n}{}' r^k = \sum_{r=1}^{r=k}{}' C_{n+1\,r+1}\, k!\, A_{(r)+k-r} = \sum_{r=1}^{r=k}{}' C_{n+1\,r+1}\, k!\, A_{(r)+k-r}.$$

Of course the integral series might, where $x = 0$, be made to depend upon Σr^k.

B. Integro-Geometric Series.

The general term of this is

$$(a_0n^k + a_1n^{k-1} + \dots a_k)x^n = x(a_0x^{n-1}n^k + a_1x^{n-1}n^{k-1} + \dots x^{n-1}a_k).$$

We may consider the sum

$$S_{nr-1} = 1^{r-1} + 2^{r-1}x + 3^{r-1}x^2 + \dots n^{r-1}x^{n-1}.$$

$$(1-x)^r S_{nr-1} = (1^{r-1} + 2^{r-1}x + 3^{r-1}x^2 + \dots n^{r-1}x^{n-1})$$

$$\left[1 - rx + \frac{r\,.\,r-1}{1\,.\,2}x^3 \dots (-1)^r x^r\right].$$

$$\begin{array}{rr|r|l|l|l|l}
=1+ & 2^{r-1} & x + 3^{r-1} & x^3 + A^{r-1} & x^3 + \dots (r-1)^{r-1} & x^{r-2} + 0 & x^{r-1} \\
 & -r & -r2^{r-1} & -r3^{r-1} & -r(r-2)^{r-1} & & \\
 & & +\frac{r\,.\,r-1}{1\,.\,2} & +\frac{r\,.\,r-1}{1\,.\,2}2^{r-1} & +\frac{r\,.\,r-1}{1\,.\,2}(r-3)^{r-1} & & \\
 & & & -\frac{r\,.\,r-1\,.\,r-2}{1\,.\,2\,3} & (-1)^{r-2}\frac{r\,.\,r-1}{1\,.\,2} & & \\
a_0 & a_1 & a_2 & a_3 & a_{r-2} & &
\end{array}$$

[$(n-r+1)$ terms from $a_{r-1}x^{r-1}$ to $a_{n-1}x^{n-1}$ vanish by virtue of I. B. b.]

$$\begin{array}{l|l|l|c}
-rn^{r-1} & x^n + \frac{r \,.\, r-1}{1.\,2}\, n^{r-1} & x^{n+1} \ldots + (-1)^{s+1}\frac{r \,.\, r-1 \ldots r-s}{1\,.\,2 \ldots s+1} & \\
\frac{r \,.\, r-1}{1\,.\,2}(n-1)^{r-1} & -\frac{r \,.\, r-1 \,.\, r-2}{1\,.\,2\,.\,3}(n-1)^{r-1} & & \begin{array}{c} x^{n+s} \\ \text{until} \\ s = r-1\,. \end{array} \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\
(-1)^r(n-r+1)^{r-1} & (-1)^r(n-r+2)^{r-1} & (-1)^r(n-r+s+1)^{r-1} &
\end{array}$$

These last terms may be written

$$x^n \left\{ \sum_{s=1}^{s=r} (-1)^s C_{rs}(n-s+1)^{r-1} + x \sum_{s=2}^{s=r} (-1)^s C_{rs}(n-s+2)^{r-1} + \ldots \right.$$

$$\left. x\lambda \sum_{s=\lambda+1}^{s=r} (-1)^s C_{rs}(n-s+\lambda+1)^{r-1} + \ldots (-1)^r n^{r-1} x^{r-1} \right\}$$

and since

$$a_\lambda = \sum_{s=0}^{s=\lambda} (-1)^s C_{rs}(\lambda - s + 1)^{r-1}$$

$$S_{nr-1} = \frac{1}{(1-x)^r} \left\{ \sum_{\lambda=0}^{\lambda=r-2} x^\lambda \sum_{s=0}^{s=\lambda} (-1)^s C_{rs}(\lambda - s + 1)^{r-1} \right.$$

$$\left. + x^n \sum_{\lambda=0}^{\lambda=r-1} x^\lambda \sum_{s=\lambda+1}^{s=r} (-1)^s C_{rs}(n - s + \lambda + 1)^{r-1} \right\}.$$

The sum of n terms of $(a_0 r^k + a_1 r^{k-1} + \ldots a_k)\, x^r$, from $r = 1$, $r = n$, is

$$a_0 x S_{nk} + a_1 x S_{nk-1} + a_2 x S_{nk-2} + \ldots a_{k-1} x S_{n1} + a_k x S_{n0}$$

and is therefore

$$x \sum_{r=0}^{r=k} a_r S_{nk-r}\,,$$

or it is

$$x \sum_{r=0}^{r=k} \frac{a_r}{(1-x)^{k-r+1}} \left\{ \sum_{\lambda=0}^{\lambda=k-r-1} x^\lambda \sum_{s=0}^{s=\lambda} (-1)^s C_{k-r+1s}(\lambda - s + 1)^{k-r} \right.$$

$$\left. + x^n \sum_{\lambda=0}^{\lambda=k-r} x^\lambda \sum_{s=\lambda+1}^{s=k-r+1} (-1)^s C_{k-r+1s}(n - s + \lambda + 1)^{k-r} \right\},$$

a complete general formula for the sum of n terms of an integro-geometric series.